Space-time presymmetries (within field theory model)

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Space–time presymmetries

H P W Gottlieb†
Mathematical Physics Department, University of Adelaide, South Australia 5001

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Abstract. Within a field theory model, the conditions under which the presymmetry subgroup of the Poincaré group may be enlarged are investigated. An equivalence to the enlargement including the dilatation operator is demonstrated.

1. The Poincaré structure and presymmetry

The persistence of certain kinematical symmetry relations, even when an external field manifestly breaking the symmetry is present, has been termed 'presymmetry' and was discussed by Ekstein (1967). It is interesting to study this type of effect within the framework of a field theory model and to investigate under what conditions the presymmetry group may be enlarged.

Specifically, we consider a spinless field ϕ and its Lagrangian density L arbitrarily depending only on ϕ and its first-order derivatives as well as explicitly depending on the space–time variable x^{μ}:

$$L = L(\phi, \partial\phi, x). \tag{1.1}$$

No assumptions of space–time symmetry are made initially; but to facilitate development using the canonical formalism, we assume the non-vanishing of the conjugate field momentum:

$$\Pi = \frac{\partial L}{\partial(\partial_0\phi)} \neq 0. \tag{1.2}$$

(The extension to several fields may readily be made.)

Tensors obtained from Noether's theorem in the case of Poincaré symmetry are now used to define the quantities

$$\Theta^{\mu\nu} = \frac{\partial L}{\partial(\partial_{\mu}\phi)}\partial^{\nu}\phi - g^{\mu\nu}L. \tag{1.3}$$

Thus

$$\Theta^{0\mu} = \Pi\, \partial^{\mu}\phi - g^{0\mu}L \tag{1.4}$$

so

$$\Theta^{0i} = \Pi\, \partial^{i}\phi \qquad i, j = 1, 2, 3. \tag{1.5}$$

† Present address: School of Science, Griffith University, Nathan 4111, Brisbane, Australia.

The momentum generators are, as usual, defined by

$$P^{\mu}=\int d^3\boldsymbol{x}\,\Theta^{0\mu} \tag{1.6}$$

and the generators of angular momentum and boosts are defined by

$$M^{\mu\nu}=\int d^3\boldsymbol{x}\,(x^{\mu}\Theta^{0\nu}-x^{\nu}\Theta^{0\mu})\equiv -M^{\nu\mu} \tag{1.7}$$

but for the present investigation we must justify their interpretation by confirming their effect on the fields, as follows.

Assume only the canonical equal-time commutation relations (ETCR) between the field ϕ and its conjugate momentum Π:

$$[\phi(\boldsymbol{x},t),\Pi(\boldsymbol{y},t)]=\mathrm{i}\,\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.8a}$$

$$[\phi(x),\phi(y)]=0=[\Pi(x),\Pi(y)]. \tag{1.8b}$$

(All commutators have equal times implied.)

Then it follows simply as a consequence of the canonical field ETCR, without assuming any space–time invariance of L, that (cf Jackiw (Treiman *et al* 1972))

$$[\Theta^{0\mu}(x),\phi(y)]=-\mathrm{i}\,\partial^{\mu}\phi\,\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.9a}$$

$$[\Theta^{0i}(x),\Pi(y)]=\mathrm{i}\Pi(x)\partial_x^i\,\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.9b}$$

$$[P^{\mu},\phi]=-\mathrm{i}\,\partial^{\mu}\phi \tag{1.10}$$

$$[M^{\mu\nu},\phi]=-\mathrm{i}(x^{\mu}\,\partial^{\nu}\phi-x^{\nu}\,\partial^{\mu}\phi) \tag{1.11}$$

justifying the nomenclature for these operators. (Note furthermore that in these and subsequent ETCR of this section the equations of motion are not invoked.)

Furthermore, one gets the condition of Schwinger (1963)

$$[\Theta^{00}(x),\Theta^{00}(y)]=-\mathrm{i}(\Theta^{i0}(x)+\Theta^{i0}(y))\,\partial_i^x\,\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.12}$$

where the ordering of superscripts is important, since here it is not necessarily true that the object $\Theta^{\mu\nu}$ is symmetric.

One also gets analogous CR (cf Boulware and Deser 1967, Deser and Morrison 1970):

$$[\Theta^{0i}(x),\Theta^{0j}(y)]=\mathrm{i}(g^{ik}\Theta^{0j}(x)+g^{jk}\Theta^{0i}(y))\,\partial_k^x\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.13a}$$

$$[\Theta^{00}(x),\Theta^{0j}(y)]=-\mathrm{i}(\Theta^{ij}(x)-g^{ij}\Theta^{00}(y))\,\partial_i^x\,\delta(\boldsymbol{x}-\boldsymbol{y})+\mathrm{i}(\partial^jL)\,\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{1.13b}$$

where ∂^jL denotes differentiation with respect to the explicit x dependence. The last commutator is thus model-dependent for general L.

These CR may be integrated to give CR for the following generators (we integrate by parts freely, assuming suitably well-behaved functions):

$$[P^i,P^j]=0 \tag{1.14}$$

$$[M^{ij},M^{kl}]=\mathrm{i}(g^{il}M^{jk}-g^{ik}M^{jl}+g^{jk}M^{il}-g^{jl}M^{ik}) \tag{1.15}$$

$$[M^{ij},P^k]=\mathrm{i}(g^{jk}P^i-g^{ik}P^j). \tag{1.16}$$

Thus, irrespective of the structure of the Lagrangian, the constructed generators realize the algebra of the three-dimensional Euclidean group E(3). This is the presymmetry group of Ekstein (1967).

As Ekstein mentions, we may compare this with the 'presymmetry group' of current algebra, which in that case is in fact the whole internal group. Even without assuming internal-group invariance, the charges constructed from the defined Noether currents realize the transformations of the fields and satisfy the group algebra. We stress that this is true not only for the case of linear representations (cf Jackiw (Treiman *et al* 1972)) but also for the case of general nonlinear realizations of internal groups (Gottlieb 1970).

Commutation relations of the other operators involve the structure of L itself to varying degrees:

$$[P^0, P^j] = \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, \partial^j L, \tag{1.17}$$

$$[M^{ij}, P^0] = \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, [(x^j\, \partial^i L - x^i\, \partial^j L) + (\Theta^{ij} - \Theta^{ji})]. \tag{1.18}$$

An enlarged presymmetry group T(4)$\boxed{\mathrm{S}}$O(3) may thus be obtained if L obeys some more restrictive condition, e.g.: (*a*) space derivatives of the quantized field ϕ must be coupled to each other in an O(3)-invariant way (so that $\Theta^{ij} = \Theta^{ji}$) and L is a function of $\boldsymbol{x}^2$ as far as explicit dependence on $\boldsymbol{x}$ is concerned, and furthermore that part of L appearing in $\partial^j L$ is an even function of $\boldsymbol{x}$ through $\phi(x)$. (*b*) $L = L_1(\phi, \partial\phi, t) + L_2(\boldsymbol{x}, t)$, L_1 O(3)-invariant. (*c*) $L = L_1(\phi, \partial\phi, t) + x_k\, \mathrm{d}^k F(\phi, t)$, L_1 O(3)-invariant.

Note that the explicit t dependence may be arbitrary.

Commutators involving the boost generators are also model-dependent in general:

$$[M^{0i}, P^j] = \mathrm{i} g^{ij} P^0 - \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, x^i\, \partial^j L. \tag{1.19}$$

Taken in conjunction with the general commutator (1.17) between P^0 and P^j, this means that L may not contain spatial $\boldsymbol{x}$ components if the Poincaré algebra is to be obtained:

$$[M^{0i}, P^0] = -\mathrm{i} P^i + \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, [-t\, \partial^i L + (\Theta^{0i} - \Theta^{i0})]. \tag{1.20}$$

For L not containing spatial $\boldsymbol{x}$ components as obtained from the commutators (1.17) and (1.19), to get this corresponding Poincaré CR, the symmetry of Θ^{0i} means that the derivatives of ϕ must be coupled to each other and in a fully Lorentz-covariant way.

$$[M^{0i}, M^{0j}] = -\mathrm{i} M^{ij} + \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, [x^j(\Theta^{i0} - \Theta^{0i}) - x^i(\Theta^{j0} - \Theta^{0j})] + \mathrm{i}t \int \mathrm{d}^3\boldsymbol{x}\, (x^j\, \partial^i L - x^i\, \partial^j L) \tag{1.21}$$

$$[M^{0i}, M^{kl}] = \mathrm{i}(g^{ik} M^{0l} - g^{il} M^{0k}) + \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, x^i(\Theta^{kl} - \Theta^{lk}) + \mathrm{i} \int \mathrm{d}^3\boldsymbol{x}\, x^i(x^l\, \partial^k L - x^k\, \partial^l L). \tag{1.22}$$

The corresponding Poincaré CR will both be satisfied if the criteria required for obtaining the previous standard CR are satisfied.

Notice also that, if $\Theta^{\mu\nu}$ is symmetric, the Poincaré CR between P^0 and P^j is satisfied if and only if that between P^0 and M^{0j} is; the Poincaré CR between M^{0i} and M^{0j} is satisfied if and only if that between P^0 and M^{ij} is.

From the preceding analyses it follows that, if the full Poincaré algebra is to be obtained, L may not contain spatial $\boldsymbol{x}$ components explicity, and the derivatives of ϕ must be coupled to each other in a Lorentz-covariant way.

L may, however, contain an otherwise arbitrary dependence on time t (cf § 4 of Ekstein 1967); it need not therefore be fully Poincaré invariant.

2. Constants of the motion

To analyse the effect of time differentiation, we now invoke for the first time the equations of motion:

$$d_\mu \frac{\partial L}{\partial \phi_{,\mu}} = \frac{\partial L}{\partial \phi} \tag{2.1}$$

where

$$d_\mu = (\partial_\mu \phi)\frac{\partial}{\partial \phi} + (\partial_\mu \partial_\nu \phi)\frac{\partial}{\partial \phi_{,\nu}} + \partial_\mu \tag{2.2}$$

where, as before, ∂_μ means differentiation with respect to explicit x dependence.

Then the equations of motion (2.1) are used to obtain, for arbitrary L,

$$d_\mu \theta^{\mu\nu} = -\partial^\nu L \tag{2.3}$$

and the time-derivative equations

$$d_0 P^\mu = -\int d^3\boldsymbol{x}\, \partial^\mu L, \tag{2.4}$$

$$d_0 M^{\mu\nu} = \int d^3\boldsymbol{x}\, [(x^\nu\, \partial^\mu L - x^\mu\, \partial^\nu L) + (\Theta^{\mu\nu} - \Theta^{\nu\mu})]. \tag{2.5}$$

Thus

$$[P^0, P^i] = -\mathrm{i}\, d^0 P^i \tag{2.6}$$

$$[P^0, M^{ij}] = -\mathrm{i}\, d^0 M^{ij} \tag{2.7}$$

and

$$d_0 P^0 = -\int d^3\boldsymbol{x}\, \partial^0 L \tag{2.8}$$

$$[P^0, M^{0j}] = \mathrm{i}P^j - \mathrm{i}\, d^0 M^{0j} + \mathrm{i}\int d^3\boldsymbol{x}\, x^j\, \partial^0 L. \tag{2.9}$$

Therefore the commutator with P^0 generates in the standard way the (total) time derivative of the presymmetry group E(3) generators. Furthermore these universal presymmetry group generators are constants of the motion ($d_0 = 0$) if and only if the enlarged presymmetry group T(4) $\boxed{\text{S}}$ O(3) CR hold.

P^0 itself is of course a constant of the motion if and only if L does not contain time t explicitly.

Finally, if the Poincaré CR between P^0 and M^{0j} holds, and if also L does not contain t explicitly, then M^{0j} is a constant of the motion.

3. The dilatation operation

We investigate the inclusion of the dilatation operation. Define the dilatation current (cf Coleman and Jackiw 1971, appendix A) by

$$\mathscr{D}^\nu(y) = y_\mu \Theta^{\nu\mu} + \frac{\partial L}{\partial \phi_{,\nu}}\phi \tag{3.1}$$

so that the dilatation generator is

$$D(t)=\int d^3y\,(y_\mu\Theta^{0\mu}+\Pi\phi). \tag{3.2}$$

Then it follows from the canonical ETCR for ϕ and Π that the fields ϕ are transformed by this constructed generator in the standard dilatation way for a field with canonical dimension one:

$$[D,\phi]=-i(x_\mu\,\partial^\mu\phi+\phi). \tag{3.3}$$

We obtain the commutators, again for arbitrary L,

$$[D,P^j]=-iP^j+t[P^0,P^j] \tag{3.4}$$

$$[D,M^{ij}]=t[P^0,M^{ij}]. \tag{3.5}$$

Therefore the presymmetry group $E(3)=T(3)\ \boxed{S}\ O(3)$ is enlarged to $T(4)\ \boxed{S}\ O(3)$ (for all t) if and only if it is enlarged to the group of E(3) and D.

Upon using the equations of motion, we obtain

$$[D,P^j]=-iP^j-it\,d^0P^j \tag{3.6}$$

$$[D,M^{ij}]=-it\,d^0M^{ij} \tag{3.7}$$

as well as the other commutators

$$[D,P^0]=-iP^0-it\,d^0P^0+i\int d^3\boldsymbol{x}\,d_\nu\mathscr{D}^\nu \tag{3.8}$$

$$[D,M^{0j}]=-it\,d^0M^{0j}-i\int d^3\boldsymbol{x}\,x^j\,d_\nu\mathscr{D}^\nu. \tag{3.9}$$

In fact, further manipulation yields

$$[D,P^0]+iP^0=i\int d^3\boldsymbol{x}\left(2\frac{\partial L}{\partial\phi_{,\mu}}\partial_\mu\phi+\frac{\partial L}{\partial\phi}\phi-x_i\,\partial^iL-4L\right). \tag{3.10}$$

Thus, if the presymmetry group is already enlarged to E(3) and D (and hence also to $T(4)\ \boxed{S}\ O(3)$, as above) it is further enlarged to $T(4)\ \boxed{S}\ O(3)$ and D if there is manifest scale invariance of ϕ, $\partial_\mu\phi$ and x_i terms (i.e. all such terms are of naive scale dimension 4). Note again that the time dependence can be arbitrary.

Similarly, we obtain

$$[D,M^{0j}]=-t([M^{0j},P^0]+iP^j)-i\int d^3\boldsymbol{x}\,x^j\!\left(2\frac{\partial L}{\partial\phi_{,\mu}}\partial_\mu\phi+\frac{\partial L}{\partial\phi}\phi-x_i\,\partial^iL-4L\right). \tag{3.11}$$

Thus if the full Poincaré algebra already holds, it is further enlarged to Poincaré and D if the integrand in (3.10) arising from the $T(4)\ \boxed{S}\ O(3)$ and D case vanishes. This integrand equals $d_\nu\mathscr{D}^\nu+t\,\partial^0L$.

Thus to obtain the enlargement of the presymmetry group to all of Poincaré and D, there must, as well as the Poincaré algebra conditions, be manifest scale invariance of the $\phi,\partial_\mu\phi$ terms (x_i already being absent explicitly); as always, the t dependence may be arbitrary.

4. Conclusions

We have analysed the conditions under which the universal presymmetry group E(3) may be enlarged, and have found relationships with the cases of constants of the motion and the inclusion of dilatations. A slightly restricted family of experiments may therefore be subject to a wider classification scheme.

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Appendix. Some converse considerations

Suppose we are given only the standard commutators (1.10), (1.11) involving the action of the generators on the fields.

(a) Then using the Jacobi identity, and assuming that the generators do not contain any c numbers or parts which may commute with ϕ (a redefinition may be required, cf Boulware and Deser 1967), we obtain the Poincaré algebra provided that all the generators are constants of the motion. Within the canonical formalism, we may then refer to § 2.

(b) If we commence with the canonical formalism of § 1, but instead of assuming the ETCR (1.8a) between ϕ and Π we start with the CR (1.10) and (1.11), then we find that any one of these latter CR, together with the vanishing of the ETCR on the left-hand side of (1.8b), implies the canonical ETCR (1.8a) (provided the space of functions $\partial^i\phi$ forms a large enough class of test functions to define the three-dimensional δ function).

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